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q-deformed oscillator associated with the Calogero model and its q-coherent state

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Abstract. We present the q-deformed para-bose oscillators associated with the (two-body) Calogero model. The q-deformed coherent state is also constructed and its resolution of unity is demonstrated.

1. Introduction

q-deformation [1] of the classical Lie algebra has been an active research area recently. For q-deformation of $su(1,1)$, with which we are concerned in this paper, in the literature one can find its realizations in terms of q-oscillators [2, 11] and q-para-bose oscillators [3]. For q-deformed $osp(1,2n)$ algebra, we refer to [4].

Very recently it was demonstrated that there is an oscillator analogy [5] for the Calogero model [6], although differential operator realization [7] of $su(1,1)$ has been known for a long time. This recently demonstrated oscillator is not the ordinary harmonic oscillator, in the sense that it contains exchange operators in the commutation relations. Therefore, it might be natural to study the q-deformation of the modified oscillator, which will expose different behaviour from that of the ordinary oscillators.

We will confine ourselves to the two-body Calogero model since this is the simplest non-trivial case. The modified oscillator arises from the relative motion part, which contains the inverse square interation. The modified oscillator is, however, not a new one: this oscillator realises para-bose algebra [8] and becomes a para-bose oscillator. This kind of oscillator has already been studied [9] with the introduction of a parity operator (exchange operator in the two-body case). It is also known that the para-bose algebra is in general isomorphic to $osp(1,2n)$ super-algebra [10]. On the other hand, q-deformation of this oscillator system has not been studied thoroughly as far as we are aware. In this paper, we will present the q-oscillator realization such that it encompasses the Calogero model as well as the para-bose oscillator and construct its coherent state with resolution of unity.

This paper is organized as follows. In section 2, we give a brief review of the q-deformation of $su(1,1)$, and we give a realization of the $su_{q^2}(1,1)$ generators associated with the (two-body) Calogero model in terms of the q-deformed modified oscillators. The realization looks the same as that in the standard q-oscillators [11] except for the appearance of the exchange operator (see (27)–(29)). These q-deformed modified oscillators provide an explicit operator realization of q-para-bose or $osp_{q^2}(1,2)$ super-algebra. A slightly different q-deformation which covers the modified oscillator has also been considered recently in [12]. In section 3, we construct the q-deformed coherent state for the para-bose oscillator. We demonstrate the resolution of unity for the q-deformed coherent state following the method of [13] given for the undeformed para-bose coherent state. This is the generalization of that of the q-deformed $osp(1,2)$ superalgebra considered in [14]. In section 4, a summary of our results and some remarks are given.

2. q-deformation of the para-bose oscillator

$su(1,1)$ satisfies the commutation relations

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0 \tag{1}$$

and the Casimir operator is given as

$$C = K_0(K_0 - 1) - K_+K_- \,. \tag{2}$$

Its q-deformation, $su_{q^2}(1,1)$, is given [2] as

$$[\tilde{K}_0, \tilde{K}_\pm] = \pm\tilde{K}_\pm \qquad [\tilde{K}_+, \tilde{K}_-] = -[2\tilde{K}_0]_{q^2} \tag{3}$$

where $[x]_\mu \equiv (\mu^x - \mu^{-x})/(\mu - \mu^{-1})$.

To find a realization of the $su_{q^2}(1,1)$ in terms of the generators of $su(1,1)$ we make an ansatz [15]:

$$\tilde{K}_0 = K_0 \qquad \tilde{K}_+ = \frac{2}{[2]_q}F(K_0)K_+ \qquad \tilde{K}_- = \frac{2}{[2]_q}K_-F(K_0)\,. \tag{4}$$

Then to satisfy the the q-deformed algebra (3), we have a recursion relation

$$\left(\frac{2}{[2]_{q^2}}\right)^2(G(K_0) - G(K_0+1)) = -[2K_0]_{q^2} \tag{5}$$

where

$$G(K_0) \equiv F^2(K_0)K_+K_- = F^2(K_0)((K_0(K_0-1) - C) \tag{6}$$

and C is the Casimir operator given in (2).

To find $G(K_0)$, we first note that $F(K_0)$ in (4) should be a non-singular operator on the Hilbert space. Therefore, for the vacuum defined as $K_-|\mathrm{vac}\rangle = 0$ and $K_0|\mathrm{vac}\rangle = k_0|\mathrm{vac}\rangle$, we have

$$G(K_0 = k_0) = 0 \tag{7}$$

to make the operator consistent. From the recursion relation (5) and the initial condition (7), we have $G(K_0) = \frac{1}{4}[2K_0 - 2k_0]_q[2K_0 + 2k_0 - 2]_q$ and this leads to the $F(K_0)$ as

$$F(K_0) = \sqrt{\frac{[2K_0 - 2k_0]_q[2K_0 + 2k_0 - 2]_q}{(2K_0 - 2k_0)(2K_0 + 2k_0 - 2)}}\,. \tag{8}$$

In this case the q-deformed Casimir operator is given as

$$\tilde{C} = [\tilde{K}_0]_{q^2}[\tilde{K}_0 - 1]_{q^2} - \tilde{K}_+\tilde{K}_- = [k_0]_{q^2}[k_0 - 1]_{q^2}\,. \tag{9}$$

Let us apply this formalism to the ordinary harmonic oscillator system. This system is described by the creation and annihilation operators $a = (x + \mathrm{i}p)/\sqrt{2}$, $a^+ = (x - \mathrm{i}p)/\sqrt{2}$. $su(1,1)$ generators are given as $K_0 = \frac{1}{2}(n + \frac{1}{2})$, $K_+ = \frac{1}{2}(a^+)^2$ and $K_- = \frac{1}{2}a^2$, where $n = a^+a$. The Casimir operator is given as $C = -\frac{3}{16}$. This system has two different representations whose k_0 is $\frac{1}{4}$ and $\frac{3}{4}$, respectively. We find that $F(K_0)$ in (8) is the same for both cases:

$$F(K_0) = \sqrt{\frac{[2K_0 - \frac{1}{2}]_q [2K_0 - \frac{3}{2}]_q}{(2K_0 - \frac{1}{2})(2K_0 - \frac{3}{2})}}\,. \tag{10}$$

Therefore, the generators of $su_{q^2}(1,1)$ are given as

$$\tilde{K}_0 = K_0 \qquad \tilde{K}_+ = \frac{2}{[2]_q}\sqrt{\frac{[n]_q[n-1]_q}{n(n-1)}}K_+ \qquad \tilde{K}_- = \frac{2}{[2]_q}K_-\sqrt{\frac{[n]_q[n-1]_q}{n(n-1)}}\,. \tag{11}$$

The q-deformed Casimir operator is given as $\tilde{C} = -[\frac{1}{4}]_{q^2}[\frac{3}{4}]_{q^2}$. One can realize the algebra $su_{q^2}(1,1)$ in terms of the q-deformed oscillator [2, 11]. From (11), we get

$$\tilde{K}_0 = \tfrac{1}{2}\big(a^+a + \tfrac{1}{2}\big) \qquad \tilde{K}_+ = \frac{1}{[2]_q}(a_q^+)^2 \qquad \tilde{K}_- = \frac{1}{[2]_q}a_q^2 \tag{12}$$

where

$$a_q = a\sqrt{\frac{[n]_q}{n}} \qquad a_q^+ = \sqrt{\frac{[n]_q}{n}}a^+\,. \tag{13}$$

For other represenatation of $su(1,1)$ we consider the relative motion part of the two-particle Calogero model. The two-particle Calogero model is given as

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{16}(x_1 - x_2)^2 + \frac{g}{(x_1 - x_2)^2}\,. \tag{14}$$

If we rescale $x_i \to \rho x_i$ and $p_i \to /(\hbar/\rho)p_i$, where $\rho = \sqrt{\hbar/m\omega}$, then we have the Hamiltonian

$$H = \hbar\omega\left[\frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{16}(x_1 - x_2)^2 + \frac{\lambda}{(x_l - x_2)^2}\right] \tag{15}$$

where $\lambda = mg/\hbar^2 \geqslant -\frac{1}{4}$. Restricting to the relative motion part only and denoting $x_1 - x_2 = 2x$ and $p_1 - p_2 = p$ such that $[x, p] = \mathrm{i}$, we have, in units of $\hbar\omega$,

$$K_0 = \frac{1}{4}\left(p^2 + x^2 + \frac{\lambda}{x^2}\right). \tag{16}$$

The other $su(1,1)$ generators are given as [7]

$$K_\pm = \frac{1}{4}\left(-p^2 + x^2 \mp \mathrm{i}(xp + px) - \frac{\lambda}{x^2}\right). \tag{17}$$

The Casimir operator has the value $\frac{1}{4}\lambda - \frac{3}{16}$, which differs from that of the harmonic oscillator system ($\lambda = 0$). Now q-deformed generators are obtained if we put

$$F(K_0) = \sqrt{\frac{\left[2K_0 - 1 - \sqrt{\frac{1}{4}+\lambda}\right]_q\left[2K_0 - 1 + \sqrt{\frac{1}{4}+\lambda}\right]_q}{\left(2K_0 - 1 - \sqrt{\frac{1}{4}+\lambda}\right)\left(2K_0 - 1 + \sqrt{\frac{1}{4}+\lambda}\right)}}\,. \tag{18}$$

For the Calogero model it is also possible to realize $su(1,1)$ in terms of a modified oscillator system by exploiting an exchange-operator formalism [5]. We will present the q-deformed version of the system, paying attention to the role of the exchange operator.

Let us introduce notations.

$$\pi = p + \mathrm{i}\frac{l}{x}M \tag{19}$$

where l is a real parameter to be determined in terms of λ in the Calogero model. M is an exchange operator in the two-body system and plays the role of the parity operator in this reduced one-body system, i.e.

$$Mp = -pM \qquad Mx = -xM \qquad M = M^+ = M^{-1} . \tag{20}$$

Next define ladder operators A and A^+ as

$$A = \frac{1}{\sqrt{2}}(x + \mathrm{i}\pi) \qquad A^+ = \frac{1}{\sqrt{2}}(x - \mathrm{i}\pi) . \tag{21}$$

Then the modified Hamiltonian is defined as

$$K_0 \equiv \frac{1}{4}(A^+A + AA^+) = \frac{1}{4}\left(p^2 + x^2 + \frac{l(l-M)}{x^2}\right) . \tag{22}$$

One may realize $su(1,1)$ algebra just as in the harmonic oscillator case by identifying

$$K_0 = \tfrac{1}{2}\left(N + \tfrac{1}{2}\right) \qquad K_+ = \tfrac{1}{2}(A^+)^2 \qquad K_- = \tfrac{1}{2}A^2 . \tag{23}$$

As in a harmonic oscillator system, we define K_0 in terms of N. The Casimir operator is given as $C = -\frac{3}{16} + \frac{1}{4}l(l-M)$. One may easily check the algebraic relations if one uses the commutation relations

$$[N, A^+] = A^+ \qquad [N, A] = -A \qquad [A, A^+] = 1 + 2lM \tag{24}$$

which are obtained from their definitions.

The Hilbert space is obtained by applying the ladder operator K_+ successively on the vacuum state which is annihilated by K_-. On the other hand, the parity operator M commutes with K_0 and $K_\pm$. Therefore, M behaves like a number as far as this $su(1,1)$ is concerned. One can obtain the eigenstates of the two-boson (two-fermion) Calogero system by restricting the Hilbert space of K_0 to a subspace of symmetric (anti-symmetric) eigenstates with λ identified as $l(l-1)$ $(l(l+1))$.

To realize the generators of $su_{q^2}(1,1)$ in (4) in terms of the q-deformed modified oscillator, we note first that k_0 for one representation can be obtained from the vacuum $|0\rangle$ defined by $A|0\rangle = 0$, which gives $k_0 = \frac{1}{4} + \frac{1}{2}lM$. The vacuum for the other reprentation, $A^+|0\rangle$ gives $k_0 = \frac{3}{4} - \frac{1}{2}lM$. Referring to (8), we have

$$F(K_0) = \sqrt{\frac{[N - lM]_q [N + lM - 1]_q}{(N - lM)(N + lM - 1)}} \tag{25}$$

independent of the representations. k_0 is no longer a number when the creation and annihilation operator A^+, A are present. The Casimir operator is given as

$$\tilde{C} = \left[\frac{1}{4} + \frac{lM}{2}\right]_{q^2}\left[-\frac{3}{4} + \frac{lM}{2}\right]_{q^2} . \tag{26}$$

Now we define the q-deformed generators in (4) in terms of q-deformed oscillators as

$$\tilde{K}_0 \equiv K_0 \qquad \tilde{K}_+ \equiv \frac{1}{[2]_q}(A_q^+)^2 \qquad \tilde{K}_- \equiv \frac{1}{[2]_q}(A_q)^2 . \tag{27}$$

Then equations (23) and (25) give

$$A_q = A\sqrt{\frac{[N - lM]_q}{N - lM}} \qquad A_q^+ = \sqrt{\frac{[N - lM]_q}{N - lM}}A^+ . \tag{28}$$

They satisfy the relations

$$\begin{aligned} &[N, A_q^+] = A_q^+ \qquad [N, A_q] = -A_q \\ &A_q^+ A_q = [N - lM]_q \qquad A_q A_q^+ = [N + 1 + lM]_q . \end{aligned} \tag{29}$$

q-deformed operators given in (27), (28) constitute a representation of q-deformed $osp(1,2)$ super-algebra. One may use this explicit form to calculate the Casimir operator of $osp_{q^2}(1,2)$:

$$\tilde{C} = [K_0]_{q^2}[K_0 - 1]_{q^2} - \tilde{K}_+\tilde{K}_- + \left[\frac{1}{2}\right]_{q^2}\left[\frac{1}{4}\right]_{q^2}\frac{\cosh\eta l}{\cosh 2\eta K_0}[A_q, A_q^+]$$
$$= \left[\frac{1}{4}\right]_{q^2}\left(2\left[\frac{1}{4}\right]_{q^2} - \left[\frac{3}{4}\right]_{q^2}\right)\cosh^2\eta l + \frac{1}{4}([2]_q + [2]_{q^2})\left(\left[\frac{l}{2}\right]_{q^2}\right)^2 \tag{30}$$

where $q = e^\eta$ and (30) reduces to $-\frac{1}{16} + \frac{1}{4}l^2$ when $q = 1$.

3. q-deformed coherent states

In this section we will construct the coherent states for the q-deformed modified oscillator obtained above and demonstrate the resolution of unity. Let us begin by summarizing the Fock space representation of the q-para-bose system. Recalling that the Fock space is unchanged under the q-deformation, we label them as $|n\rangle, n = 0, 1, 2, \ldots$. The vacuum $|0\rangle$ is defined by

$$A|0\rangle = 0 \qquad M|0\rangle = |0\rangle \tag{31}$$

and therefore, from (22)–(24),

$$K_0|0\rangle = \beta|0\rangle \qquad \beta \equiv \frac{1}{4} + \frac{l}{2} \qquad N|0\rangle = l|0\rangle\,. \tag{32}$$

Since K_0 is positive-definite, it is clear that $l \geqslant -\frac{1}{2}$. The orthonomalized ket $|n\rangle$ is given as

$$|n\rangle = \frac{1}{\sqrt{C_n!}}A^{+n}|0\rangle = \frac{1}{\sqrt{\tilde{C}_n!}}A_q^{+n}|0\rangle \tag{33}$$

where

$$C_n = n + l(1 - (-1)^n) \qquad \tilde{C}_n = [C_n]_q \tag{34}$$

and we use a notation $f_n! \equiv \Pi_{i=1}^n f_i$, and $0! = 1$. We also note for later convenience,

$$A_q^+|n\rangle = \sqrt{\tilde{C}_{n+1}}|n+1\rangle \qquad A_q|n\rangle = \sqrt{\tilde{C}_n}|n-1\rangle \tag{35}$$
$$N|n\rangle = (n+l)|n\rangle \qquad M|n\rangle = (-1)^n|n\rangle\,. \tag{36}$$

We recall that the set $\{|n\rangle\}$ forms a single irreducible representation of $osp(1,2)$ algebra with Casimir $C(osp(1,2)) = -\frac{1}{16} + \frac{1}{4}l^2$. On the other hand, $\{|2p\rangle\}$ and $\{|2p+1\rangle\}$ for $p = 0, 1, 2, \ldots$ form two distinct, parity-even and -odd irreducible representations D_α and $D_{\beta+\frac{1}{2}}$ of $su(1,1)$ algebra, respectively. We will label the eigenstates as $|k, p\rangle$:

$$|k,p\rangle = \left\{\begin{array}{ll} |2p\rangle & k = \beta \\ |2p+1\rangle & k = \beta + \frac{1}{2} \end{array}\right\} \qquad K_0|k,p\rangle = (k+p)|k,p\rangle\,. \tag{37}$$

It is simple to show from (23), (27), and (33)–(37),

$$|k,p\rangle = \frac{1}{\sqrt{d_p!}}K_+^p|k,0\rangle = \frac{1}{\sqrt{\tilde{d}_p!}}\tilde{K}_+^p|k,0\rangle$$
$$\tilde{K}_+|k,p\rangle = \sqrt{\tilde{d}_{p+1}}|k,p+1\rangle \qquad \tilde{K}_-|k,p\rangle = \sqrt{\tilde{d}_p}|k,p-1\rangle\,. \tag{38}$$

where

$$d_p = p(p+2k-1) \qquad \tilde{d}_p = [p]_{q^2}[p+2k-1]_{q^2}\,. \tag{39}$$

We will use the definition of the unnormalized q-coherent state $|z)$ for the q-para-bose oscillator as

$$A_q|z) = z|z) \tag{40}$$

where z is a complex number. Then $|z)$ is given by

$$|z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\tilde{C}_n!}}|n\rangle = \left\{\sum_{n=0}^{\infty} \frac{z^n}{\tilde{C}_n!}(A_q^+)^n\right\}|0\rangle\,. \tag{41}$$

We can associate with each normalizable ket $|\phi\rangle$,

$$|\phi\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\phi\rangle \equiv \sum_{n=0}^{\infty} \phi_n|n\rangle \tag{42}$$

an entire function $\phi(z)$ defined as

$$\phi(z) \equiv (z^*|\phi\rangle = \sum_{n=0}^{\infty} \frac{\phi_n}{\sqrt{\widetilde{C}_n!}} z^n\,. \tag{43}$$

Combining (40) and (43), we get

$$(z^*|A_q^+|\phi\rangle = z(z^*|\phi\rangle = z\phi(z) \tag{44}$$

which shows that A_q^+ acts as a multiplication by z in the Bargmann space. Taking $|\phi\rangle$ as $A_q|\psi\rangle$ in (44), and using (29),

$$z(z^*|A_q|\psi\rangle = (z^*|A_q^+A_q|\psi\rangle = (z^*|[N-lM]_q|\psi\rangle\,. \tag{45}$$

Using (34), (36), and (43), equation (45) shows that A_q acts as an operator ($q = \mathrm{e}^{\eta}$)

$$A_q \to (\cosh^2\eta l + M\sinh^2\eta l)\frac{\mathrm{d}}{\mathrm{d}(z;q)} + \frac{[2l]_q}{4z}(T_q + T_q^{-1})(1-M)\,. \tag{46}$$

Here, the parity operator M acts as

$$M\psi(z) = \psi(-z) \tag{47}$$

and the q-derivative and q-shift operators are given as usual by

$$T_q\psi(z) = \psi(qz) \tag{48}$$

$$\frac{\mathrm{d}}{\mathrm{d}(z;q)}\psi(z) = \frac{1}{z}\frac{T_q - T_q^{-1}}{(q-q^{-1})}\psi(z)\,. \tag{49}$$

We need a resolution of unity to complete the q-coherent state descriptions. It will provide a natural inner product for the Bargmann space, under which A_q and A_q^+ are Hermitian conjugates mutually. To this end, we proceed in an analogous way to that taken in [13] for the para-bose coherent states. The essence of the method lies in finding the correct measure of $|z)$ coherent states by using those of the $su(1,1)$ coherent states.

Let us introduce the $su_{q^2}(1,1)$ coherent states as in [13, 16] (for further discussions on $su_q(2)$, $su_q(1,1)$ coherent states, see [17]):

$$|\omega;k) \equiv \sum_{p=0}^{\infty} \frac{1}{\sqrt{\widetilde{d}_p!}}\omega^p|k,p\rangle = \sum_{p=0}^{\infty} \frac{1}{\widetilde{d}_p!}\omega^p \widetilde{K}_+^p|k,0\rangle \tag{50}$$

where ω is a complex number and other notations are given in (37)–(39) and satisfy

$$\tilde{K}_-|\omega; k) = \omega|\omega; k) \,. \tag{51}$$

For a general ket $|g\rangle$ in D_k,

$$|g\rangle = \sum_{p=0}^{\infty} |k, p\rangle\langle k, p|g\rangle = \sum_{p=0}^{\infty} g_p|k, p\rangle \tag{52}$$

we may associate a complex function $g(\omega)$ defined as

$$g(\omega) \equiv (\omega^*; k|g\rangle = \sum_{p=0}^{\infty} \frac{g_p}{\sqrt{\widetilde{d}_p!}}\omega^p \,. \tag{53}$$

The action of $su_{q^2}(1,1)$ generators on $g(\omega)$ is represented as

$$K_0 \to k + \omega\frac{\mathrm{d}}{\mathrm{d}\omega} \tag{54}$$

$$\tilde{K}_+ \to \omega \qquad \tilde{K}_- \to \left\{ [2k]_{q^2}\frac{T_{q^2} + T_{q^2}^{-1}}{2} + \omega \cosh 4\eta k \frac{\mathrm{d}}{\mathrm{d}(\omega; q^2)} \right\} \frac{\mathrm{d}}{\mathrm{d}(\omega; q^2)} \,. \tag{55}$$

The q-coherent state $|z)$ in (41) can be written in terms of $|\omega; \beta)$ and $|\omega; \beta + \frac{1}{2})$ compactly as

$$|z) = |\omega; \beta) + \frac{z}{\sqrt{[4\beta]_q}}|\omega; \beta + \tfrac{1}{2}) \tag{56}$$

with ω being identified as

$$\omega = \frac{z^2}{[2]_q} \tag{57}$$

with the help of (33)–(39). Therefore, $\phi(z)$ in (43) is decomposed into even and odd parts:

$$\begin{aligned} &\phi(z) = \phi_+(z) + \phi_-(z) \\ &\phi_+(z) = (\omega^*; \beta|\phi\rangle \equiv \phi_1(\omega) \\ &\phi_-(z) = \frac{z}{\sqrt{[4\beta]_q}}\left(\omega^*; \beta + \tfrac{1}{2}|\phi\right\rangle \equiv \frac{z}{\sqrt{[4\beta]_q}}\phi_2(\omega) \,. \end{aligned} \tag{58}$$

We remark in passing that if we express $\phi(z)$ as a column vector

$$\begin{pmatrix} \phi_+(z) \\ \phi_-(z) \end{pmatrix}$$

then the operators are realized by matrix operators as follows (remembering that A_q and A_q^+ change the parity):

$$A_q^+ \to \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix} \tag{59}$$

$$A_q \to \begin{pmatrix} 0 & \frac{1}{2}(q^{2l} + q^{-2l})\dfrac{\mathrm{d}}{\mathrm{d}(z;q)} + \dfrac{[2l]_q}{2z}(T_q + T_q^{-1}) \\ \dfrac{\mathrm{d}}{\mathrm{d}(z;q)} & 0 \end{pmatrix} . \tag{60}$$

$\tilde{K}_\pm$ and K_0 are given as a diagonal matrix in this basis and their explicit form can be obtained using the relations in (27). A similar consideration has been done in [18] especially in connection with q-special functions.

To demonstate the resolution of unity for $|z)$, suppose we have obtained a resolution of unity for $|\omega; k)(k = \beta$ or $\beta + \frac{1}{2})$ coherent states in (50), namely

$$\int d^2(\omega; q^2) G_k(|\omega|)|\omega; k)(\omega; k| = \sum_{p=0}^{\infty} |k, p\rangle\langle k, p| = I \tag{61}$$

where

$$d^2(\omega; q^2) \equiv \tfrac{1}{2} d(|\omega|^2; q^2) d\theta = \frac{[2]_q}{2} |\omega| d(|\omega|; q) d\theta \tag{62}$$

and $d(|\omega|; q)$ is a standard q-integration [19, 20] and $d\theta$ is an ordinary integration from 0 to 2π. Then, using the definition of an inner product in D_k,

$$\langle g'|g\rangle = \sum_{p=0}^{\infty} g_p'^* g_p = \int d^2(\omega; q^2) G_k(|\omega|) g'^*(\omega) g(\omega) \tag{63}$$

where $g(\omega), g'(\omega)$ are associated with $|g\rangle$, $|g'\rangle$ as in (52)–(53), we have

$$\langle \phi'|\phi\rangle = \int d^2(\omega; q^2) \left\{ G_\beta(|\omega|) \phi_1'^*(\omega) \phi_1(\omega) + G_{\beta+\frac{1}{2}}(|\omega|) \phi_2'^*(\omega) \phi_2(\omega) \right\} \tag{64}$$

for arbitrary two kets $|\phi\rangle$ and $|\phi'\rangle$ in the total Fock space by referring to (37) and (58). In equation (64), we make the change of variable as in (57) and take into account that ω covers the complex plane twice while z covers it once:

$$d^2(\omega; q^2) \to \frac{|z|^2}{[2]_q} d^2(z; q). \tag{65}$$

Collecting (58), (64) and (65) and observing $|\phi\rangle, |\phi'\rangle$ are arbitrary, we end up with the resolution of unity for $|z)$ coherent states:

$$\begin{aligned} \frac{1}{2[2]_q} \int d^2(z; q) \Bigg[\left\{ |z|^2 G_\beta\left(\frac{|z|^2}{[2]_q}\right) + [4\beta]_q G_{\beta+\frac{1}{2}}\left(\frac{|z|^2}{[2]_q}\right) \right\} |z)(z| \\ + \left\{ |z|^2 G_\beta\left(\frac{|z|^2}{[2]_q}\right) - [4\beta]_q G_{\beta+\frac{1}{2}}\left(\frac{|z|^2}{[2]_q}\right) \right\} |z)(-z| \Bigg] = I. \end{aligned} \tag{66}$$

Thus, the problem to prove the resolution of unity for $|z)$ coherent states reduces to finding $G_k(|\omega|)$ satisfying (61).

The diagonal element of (61) in $|k, p\rangle$ basis gives

$$\int d(u; q) u^{2p+1} G_k(u) = \frac{1}{\pi [2]_q} \prod_{j=1}^{p} ([j]_{q^2} [j + 2k - 1]_{q^2}) \tag{67}$$

where we have put $|\omega| = u$, and θ-integration is done. To find $G_k(u)$, we recall the undeformed one, $G_k^{(0)}(u)$ [13],

$$G_k^{(0)}(u) = \frac{2u^{2k-1} K_{2k-1}(2u)}{\pi \Gamma(2k)} \tag{68}$$

where $K_\nu(x)$ is a modified Bessel function.

For the q-deformed case, let us consider first the simplest $\beta = \frac{1}{4}(l = 0)$. In this case we need $G_{\frac{1}{4}}(u)$ and $G_{\frac{3}{4}}(u)$. For $k = \frac{1}{4}$, equation (67) becomes

$$\int d(u; q) u^{2p+1} G_{\frac{1}{4}}(u) = \frac{1}{\pi [2]_q} \prod_{j=1}^{p} ([j]_{q^2} [j - \tfrac{1}{2}]_{q^2}) = \frac{[2p]_q!}{([2]_q)^{2p+1} \pi} \tag{69}$$

where we use $[2x]_q = [2]_q[x]_{q^2}$ and $[2p]_q! \equiv \prod_{j=1}^{2p}[j]_q$. To solve this, we employ the q-exponential function defined as

$$e_q(\upsilon) \equiv \begin{cases} \sum_{n=0}^{\infty} \frac{\upsilon^n}{[n]_q!} & \text{for } \upsilon > -\zeta \\ 0 & \text{otherwise} \end{cases} \tag{70}$$

where $-\zeta$ is the largest zero of $e_q(\upsilon)$. The integration representation of the q-factorial is given as

$$\int_0^{\infty} \mathrm{d}(w; q) e_q(-w) w^m = [m]_q! \,. \tag{71}$$

From this we get

$$G_{1/4}(u) = \frac{1}{\pi}\frac{1}{u} e_q(-[2]_q u) \,. \tag{72}$$

Likewise for $k = \frac{3}{4}$, we have

$$G_{3/4}(u) = \frac{[2]_q}{\pi} e_q(-[2]_q u) \,. \tag{73}$$

The resolution of unity, equation (66) is reduced to that of the q-coherent states of the q-oscillator [19, 20] in this case:

$$\int \frac{\mathrm{d}^2(z; q)}{\pi} e_q(-|z|^2)|z)(z| = I \,. \tag{74}$$

In fact, $|z)$ coincides with the usual q-coherent states as can be seen in (34) and (41),

$$|z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_q!}} |n\rangle \,. \tag{75}$$

This corresponds to the result of [14], 'two-component' coherent states representations of the q-deformed $osp(1, 2)$ superalgebra realized in terms of the q-oscillators ($|\omega; k = \frac{1}{4})$ and $z|\omega; k = \frac{3}{4})$ are denoted as $||z\rangle_1$ and $||z\rangle_2$, respectively).

Let us consider case of para-bose oscillator with $l = 1, 2, 3 \ldots$. Then we may put $2k - 1 = n + \frac{1}{2}$ such that $n = 0, 1, 2, \ldots$ in (67). We rewrite the (67) as

$$\int \mathrm{d}(u; q) u^{2p+1} G_k(u) = \frac{1}{\pi([2]_q)^{2p+2}[2n+1]_q!}[2n + 2p + 2]_q! \frac{[p]_{q^2}![n]_{q^2}!}{[n + 1 + p]_{q^2}!} \,. \tag{76}$$

To find $G_k(u)$ we use the integration representation of the q-beta function:

$$\int_{\alpha^n}^{\infty} \mathrm{d}(x; \alpha) \frac{(x-1)_\alpha^n}{x^{n+p+2}} = \frac{[n]_\alpha![p]_\alpha!}{[n + 1 + p]_\alpha!} \tag{77}$$

where

$$(\upsilon + \omega)_\alpha^n \equiv \sum_{m=0}^{n} \frac{[n]_\alpha!}{[m]_\alpha![n-m]_\alpha!} \upsilon^{n-m} \omega^m \,. \tag{78}$$

Now using the formulae for q-factorial and q-beta we have $G_k(u)$ as

$$G_{\frac{3}{4}+\frac{n}{2}}(u) = \frac{[2]_q}{\pi^{3/2} \prod_{j=0}^{n} [j + \frac{1}{2}]_{q^2}} u^{n+1/2} \widetilde{K}_{n+\frac{1}{2}}([2]_q u) \tag{79}$$

where $\tilde{K}_{n+1/2}(x)$ is the q-deformed modified Bessel function in integration representtion for half-odd integer ν:

$$\tilde{K}_{n+\frac{1}{2}}(x) = \left(\frac{x}{[2]_q}\right)^{n+1/2} \frac{\sqrt{\pi}}{[n]_{q^2}!} \int_{q^n}^{\infty} \mathrm{d}(t;q) e_q(-xt)(t^2-1)^n_{q^2}\,. \tag{80}$$

Recalling $\beta = \frac{1}{4} + \frac{1}{2}l$, for $l = n+1 = 1, 2, 3, \ldots$, we have the resolution of unity in (66) written as

$$I = \frac{1}{[2]_q^{n+\frac{1}{2}} \prod_{j=0}^{n}[j+\frac{1}{2}]_{q^2}} \int \frac{\mathrm{d}^2(z;q)}{2\pi} |z|^{2n+3} \Big[\{\tilde{K}_{n+\frac{1}{2}}(|z|^2) + \tilde{K}_{n+\frac{3}{2}}(|z|^2)\}|z)(z| \\ + \{\tilde{K}_{n+\frac{1}{2}}(|z|^2) - \tilde{K}_{n+\frac{3}{2}}(|z|^2)\}|z)(-z|\Big]\,. \tag{81}$$

Similarly for the case with $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$, we put $2k-1 = n$ where $n = 0, 1, 2, \ldots$ in (67). We may rewrite (67) as

$$\int \mathrm{d}(u;q) u^{2p+1} G_k(u) = \frac{[p]_{q^2}![p+n]_{q^2}!}{\pi [2]_q [n]_{q^2}!}\,. \tag{82}$$

Using the integration representation of the q-factorial in (71) twice, we have $G_k(u)$ as

$$G_{(n+1)/2}(u) = \frac{[2]_q}{\pi [n]_{q^2}!} u^n \tilde{K}_n([2]_q u) \tag{83}$$

where $\tilde{K}_n(x)$ is the q-deformed modified Bessel function in integration representtion for integer ν:

$$\tilde{K}_n(x) = \frac{1}{[2]_q}\left(\frac{x}{[2]_q}\right)^n \int_0^{\infty} \mathrm{d}(t;q^2) \frac{1}{t^{n+1}} e_{q^2}(-t)\; e_{q^2}\left(-\frac{x^2}{([2]_q)^2 t}\right)\,. \tag{84}$$

Now the resolution of unity is written as

$$I = \frac{1}{([2]_q)^n [n]_{q^2}!} \int \frac{\mathrm{d}^2(z;q)}{2\pi} |z|^{2n+2} \\ \times \left[\{\tilde{K}_n(|z|^2) + \tilde{K}_{n+1}(|z|^2)\}|z)(z| + \{\tilde{K}_n(|z|^2) - \tilde{K}_{n+1}(|z|^2)\}|z)(-z|\right]\,. \tag{85}$$

We may normalize the q-coherent states $|\omega; k)$ as

$$|\omega; k\rangle = \frac{|\omega; k)}{N_k(|\omega|^2)} \tag{86}$$

where, using (50) and (39),

$$N_k^2(|\omega|^2) = (\omega; k|\omega; k) = \sum_{p=0}^{\infty} \frac{|\omega|^{2p}}{[p]_{q^2}! \prod_{j=1}^{p}[j+2k-1]_{q^2}!} \tag{87}$$

which is identified as a q-hypergeometric function. Note also that

$$\langle\omega'; \beta|\omega; \beta+\tfrac{1}{2}\rangle = 0 \qquad \langle\omega'; k|\omega; k\rangle = \frac{N_k^2(\omega'^*\omega)}{N_k(|\omega'|^2) N_k(|\omega|^2)} \tag{88}$$

showing that the q-coherent states are not orthogonal in each D_β or $D_{\beta+\frac{1}{2}}$. Using the above equations (86)–(88) and (56)–(57), the normalized $|z\rangle$ is given as

$$|z\rangle = \frac{|z)}{\mathcal{N}(|z|^2)}$$
$$\mathcal{N}^2(|z|^2) = (z|z) = N_\beta^2(|\omega|^2) + \frac{|z|^2}{[4\beta]_q} N_{\beta+\frac{1}{2}}^2(|\omega|^2)$$
$$\langle z'|z\rangle = \frac{\mathcal{N}^2(z'^* z)}{\mathcal{N}(|z'|^2)\mathcal{N}(|z|^2)} . \tag{89}$$

4. Summary and remarks

We have found a q-deformed version of para-bose oscillator associated with the two-body Calogero model. It realizes the $su_{q^2}(1,1)$ algebra whose form looks the same as in the standard q-oscillator case. We especially note that the q-oscillator A_q and A_q^+ are invariant under $q \to q^{-1}$, in contrast with those considered in [12], while they share the same Fock space.

The q-coherent states of the para-bose oscillators are also constructed and the resolution of unity is demonstrated for order $2l+1 = 1, 2, 3 \ldots$ employing the integration representation of the q-deformed exponential and q-deformed beta function. For non-integral values of $2l+1$, we note that the proof for the resolution of unity may need carefully developed q-special functions.

Also, we remark that for more than the two-body Calogero model, the q-deformation does not go parallel with that of the two-body case. The method given in the text is not generalized straightforwardly to the many-body case. In addition, the equivalent oscillators for the $N(\geqslant 3)$-body Calogero model [5] do not satisfy the para-bose algebra. Therefore, the q-deformation for the many-body case seems quite a challenging problem.

Note added. When we finished this article, A J Macfarlane sent us the paper [21] which interprets the modified oscillator (Calogero–Vasiliev oscillator) as the para-bosonic one, as we pointed out in section 2.

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6. References

[1] Drinfeld V G 1986 *Proc. ICM* (Berkeley) p 798
Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[2] Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415
Ui H and Aizawa N 1990 *Mod. Phys. Lett.* A **5** 237
[3] Floreanini R and Vinet L 1990 *J. Phys. A: Math. Gen.* **23** L1019
Odaka K, Kishi T and Kamefuchi S 1991 *J. Phys. A: Math. Gen.* **24** L591
[4] Floreanini R, Spiridonov V P and Vinet L 1990 *Phys. Lett.* **242B** 383
Palev T D 1993 *J. Phys. A: Math. Gen.* **26** L1111
[5] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703
Brink L, Hansson T H and Vasiliev M A 1992 *Phys. Lett.* **286B** 109

Brink L, Hansson T H, Konstein S and Vasiliev M A 1993 *Nucl. Phys.* B **401** 591
[6] Calogero F 1971 *J. Math. Phys.* **12** 419
Gambardella P J 1975 *J. Math. Phys.* **16** 1172
Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313
[7] Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)
[8] Green H S 1953 *Phys. Rev.* **90** 270
Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Tokyo: University of Tokyo Press) (Berlin: Springer)
[9] Mukunda N, Sudarshan E C G, Sharma J K and Mehta C L 1980 *J. Math. Phys.* **21** 2386
Ohnuki Y and Kamefuchi S 1978 *J. Math. Phys.* **19** 67
[10] Ganchev A Ch and Palev T D 1980 *J. Math. Phys.* **21** 797
[11] Macfarlane A J 1989 *J. Phys. A : Math. Gen.* **22** 4581
Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[12] Brzezinski T, Egusquiza I L and Macfarlane A J 1993 *Phys. Lett.* **311B** 202
[13] Sharma J K, Mehta C L, Mukunda N and Sudarshan E C G 1981 *J. Math. Phys.* **22** 78
[14] Kuang L-M, Zeng G-J and Wang F-B 1993 *J. Phys. A: Math. Gen.* **26** 4011
[15] Cutright T L and Zachos C K 1990 *Phys. Lett.* **243B** 237
[16] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[17] Chaichian M, Ellinas D and Kulish P 1990 *Phys. Rev. Lett.* **65** 980
Ellinas D 1993 *J. Phys. A: Math. Gen.* **26** L543
Quesne C 1991 *Phys. Lett.* **153A** 303
Gong R 1992 *J. Phys. A: Math. Gen.* **25** L1145
[18] Floreanini R and Vinet L 1992 *Phys. Lett.* **277B** 442
[19] Gray R W and Nelson C A 1990 *J. Phys. A: Math. Gen.* **23** L945
[20] Bracken A J, McAnally D S, Zhang R B and Gould M D 1991 *J. Phys. A: Math. Gen.* **24** 1379
[21] Macfarlane A J 1993 Generalised oscillator systems and their para-bosonic interpretation *Preprint* DAMTP 93-37